G-4/428/21

Roll No. ....

M.Sc. IV Semester Examination, 2021

MATHEMATICS

Paper I

(Functional Analysis–II)

Time : 3 Hours] [Max. Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections. Section A is objective type/multiple choice questions with no internal choice. Section B is short answer type with internal choice. Section C is long answer type with internal choice.*

SECTION A 1×10=10

(Objective Type Questions)

Choose the correct answer :

1. Norm is a :

(a) Continuous function

(b) Bicontiuous function

(c) Linear function

(d) None of these.

2. l_p, $1 \leq p < \infty$ is a :

(a) linear space (b) Non-linear space

(c) Reflexive space

(d) Linear and reflexive space

P.T.O.

3. If T^{-1} is continuous, then T is :

(a) T^{-1} is open (b) T is open

(c) T^{-1} is closed (d) T is open and closed.

4. If $p(x) = \| f \| \cdot \| x \|$, then p is :

(a) Linear functional

(b) Bounded functional

(c) Bounded and linear

(d) Sublinear functional

5. If X is an inner-product space and $x, y \in X$, then the inequality $|\langle x, y\rangle| \leq \| x \| \cdot \| y \|$ is said to be :

(a) Triangle inequality

(b) Cauchy-Schwarz inequality

(c) Triangle and Cauchy-Schwarz inequality

(d) None of these.

6. Let $T \in B(X)$ and $T = T^*$, then T is :

(a) Normal (b) Hermitian

(c) Unitary (d) None of these.

7. Dual of reflexive space is :

(a) Reflexive (b) Closed

(c) Open (d) Not reflexive.

8. A subset $A \subset X$ is said to be relatively compact, if

(a) A is closed in X (b) A is compact in X

(c) $\bar{A}$ is compact in X (d) $\bar{A}$ is closed in X.

9. Let $T \in B(X, Y)$ and $T^* : Y^* \to X^*$ s.t. $T^*(y^*)(x) = y^*(T(x))$. Then T is :

(a) Linear (b) Bounded

(c) Linear and bounded (d) None of these.

10. If $T \in B(X)$ is an isometric isomorphism on X, then T is :

(a) Normal (b) Positive

(c) Hermitian (d) Unitary.

SECTION B **4×5=20**

(Short Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit-I

1. State and prove closed graph theorem.

Or

Show that the principle of Uniform boundedness is not valid if X is not complete.

Unit-II

2. Let E be a normed linear space over K and let S be a linear subspace of E. Suppose $z \in E$ and dist $(z, S) = d > 0$. Then there exists $g \in E^*$ such that $g(S) = \{0\}$, $g(z) = d$ and $\| g \| = 1$.

Or

Let X and Y be normed spaces and $F : X \to Y$ be linear. Then F is compact if and only if for every bounded sequence $\{x_n\}$ in X, $\{F(x_n)\}$ has a subsequence which converges in Y.

Unit-III

3. Define sequential continuity. Show that the inner product is a continuous function.

Or

Show that $\mathbb{C}^n$ is a complex inner-product space.

Unit-IV

4. State and prove Pythagorean theorem of an inner-product space.

Or

Let X be a separable infinite dimensional complex Hilbert space. Then X is isometrically isomorphic to l_2.

Unit-V

5. The set of all self adjoint operators is a closed real linear subspace of $B(X)$.

Or

If $T \in B(X)$ is such that $\langle Tx, x \rangle = 0$ for all $x \in X$, then $T = 0$.

SECTION C 10×5=50

(Long Answer Type Questions)

Note : Attempt all questions.

Unit-I

1. Let X be a Banach space and let Y be a normed space over the field K. If a set F of bounded linear transformations from X into Y is pointwise bounded, then it is uniformly bounded.

Or

A bounded linear transformation T from a Banach space X onto a Banach space Y has the property that the image $T(B_0)$ of the open unit ball $B_0 = B(0, 1) \subset X$ contains an open ball about $0 \in Y$.

Unit-II

2. Let E be a complex normed linear space and let

M be a linear subspace of E. If $f \in M^*$, then there exists $g \in E^*$ such that $f \subset g$ and $\| g \| = \| f \|$.

Or

Let X and Y are normed spaces and $F \in B(X, Y)$. Then

(a) $R(F) \subset \{y \in Y : y^*(y) = 0 \text{ for all } y^* \in Z(F^*)\}$. Equality holds iff $R(F)$ is closed in Y.

(b) $R(F^*) \subset \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in Z(F)\}$. Equality holds if X and Y are Banach spaces and $R(F)$ is closed in Y.

Unit-III

3. State and prove Jordan-von Neumann theorem.

Or

Let $\{e_i\}$ be a nonvoid arbitrary orthonormal set in an inner-product space X. Then following four conditions are equivalent :

(i) $\{e_i\}$ is complete

(ii) $x \perp \{e_i\} \Rightarrow x = 0$

(iii) $x \in X \Rightarrow x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$

(iv) $x \in X \Rightarrow \| x \|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$.

Unit-IV

4. Show that every Hilbert space is reflexive.

Or

If M is a closed subspace of a Hilbert space X and $x \in X$, then there exists unique element y in M and z in $M^{\perp}$ such that $x = y + z$.

Unit-V

5. State and prove Lax-Miligram theorem.

Or

If $T \in B(X)$ is normal if and only if its real and imaginary parts commute.

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